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# **Emittance Growth through a Betatron Instability Decoherence**

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# EMITTANCE GROWTH THROUGH A BETATRON INSTABILITY DECOHERENCE

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This note outlines briefly how a self-stabilizing coherent betatron instability while saturating leads to a constant rate of emittance growth. A simple model calculation done for a slow head-tail instability existing in the Tevatron yields the emittance growth of  $10^{-13}$  m rad sec<sup>-1</sup>.

## 1. Introduction

A systematic analysis<sup>1</sup> of the coherent betatron instability in the Tevatron's fixed target mode reveals existence of a rapidly growing coupled-bunch head-tail instability driven by a combined transverse impedance of the resistive wall and kicker magnets. Following suggestions of Dave Finley one may expect that some relic of this instability with much longer growth time is also present when only a few bunches of higher intensity and larger longitudinal emittance are injected into the machine e.g. the colliding beam mode. We show here that a spontaneously "boot-strapping" betatron instability loses its initial exponential character, after a while, due to amplitude-dependent decoherence (Landau damping) and saturates to a state of asymptotically constant rate of emittance growth. A simple analytic formula describing the emittance growth is derived for a head-tail instability suppressed by the octupole-induced tune spread and is illustrated by a numerical example.

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<sup>1</sup> "Coherent Betatron Instability in the Tevatron" - S.A. Bogacz, M. Harrison and K.Y. Ng, to be published

## 2. Self-stabilizing Coherent Mode

Time evolution of the coherent betatron instability is governed by the following equation of motion for its amplitude,  $x$

$$\dot{x} = \frac{1}{\tau} x D(x). \quad (1)$$

Here  $\tau$  is the initial exponential growth-time,  $x$  is the largest betatron amplitude within the bunch and  $D(x)$  describes a specific decoherence process, which can be understood as follows.

Treating a beam as an ensemble of betatron oscillators, with some distribution of resonant frequencies, one can visualize coherent betatron motion using the following single particle picture. Each oscillator is driven by the coherent wake force of the beam, which increases exponentially with the rate  $1/\tau$ . If the resonant frequency of the oscillator departs from the frequency of the driving force by less than  $1/\tau$  its response is still "coherent". If the resonant frequency shift is larger than  $1/\tau$  for a particular oscillator it no longer participates in the coherent motion, therefore its contribution to the driving force 'drops out'. This simple selection rule is the key point of the so called Landau damping mechanism. In the above context the coherence factor  $D(x)$  is defined as a ratio of the number of "coherent" oscillators to the total number of particles in the beam.

Two dominant sources of decohering tune spread due to nonlinearities can be identified as finite chromaticity effects (sextupoles) and amplitude-dependent betatron tune (induced by the presence of the octupole field). The first mechanism is intrinsically built into the treatment of the coherent head-tail instability, Ref. 1, and manifests itself through the explicit chromaticity-dependence of the instability growth-time. Therefore, one needs to include, at this point, only the second contribution; i.e. the amplitude-dependence of the betatron tune, which can be described as follows

$$\Delta\nu = \frac{G}{\pi} \varepsilon, \quad (2)$$

where

$$G = \frac{3}{8} \frac{1}{2\pi} \beta^2 \sum_{\text{magn.}} \frac{B_0}{B\rho} \int ds b_3(s). \quad (3)$$

Here,  $B_0$  and  $b_3$  characterize the octupole field component,  $1/B\rho$  is the magnetic rigidity and  $\beta$  is the  $\beta$ -lattice function. The transverse emittance,  $\epsilon$ , is introduced as the Courant-Snyder invariant multiplied by  $\pi$

$$\epsilon = \frac{\pi x^2}{\beta}. \quad (4)$$

One can translate previously described critical tune shift into a critical betatron amplitude,  $a$ , expressed by

$$a = \sqrt{\frac{\beta}{G\omega_0\tau}}, \quad (5)$$

where  $\omega_0$  is the revolution frequency.

Making use of Sacherer's model of the head-tail instability, which postulates particular statistical distribution of particles with a given betatron amplitude as it varies along the bunch, one can construct explicitly the decoherence factor  $D(x)$  as follows

$$D(x) = \begin{cases} 1 & x < a \\ 1 - \sqrt{1 - a^2/x^2} & x \geq a. \end{cases} \quad (6)$$

Now the problem of time evolution of the decohering head-tail mode reduces to a self-consistent solution of Eqs. (1) and (6). Its long time behavior can be easily inferred by integrating Eq. (1) with the following asymptotic form of  $D(x)$

$$D(x) \approx \frac{a^2}{2x^2} \quad x \gg a. \quad (7)$$

The asymptotic equation of motion is now given by

$$\dot{x} = \frac{a^2}{2\tau x}. \quad (8)$$

one can immediately rewrite Eq. (8) as follows

$$\dot{\epsilon} = \frac{\pi a^2}{\beta \tau} . \quad (9)$$

Finally, eliminating the critical amplitude,  $a$ , given by Eq. (5) from the above formula yields the following expression for the constant rate of of emittance growth

$$\dot{\epsilon} = \frac{\pi}{G \omega_0 \tau^2} . \quad (10)$$

### 3. Emittance Growth - Numerical Example

Let us consider six equally spaced proton bunches of intensity  $6 \times 10^{10}$  ppb at 900 GeV. The bunch-length is described by  $\sigma$  of 0.4m and the horizontal betatron tune is  $\nu_x = 0.405$ . Assuming chromaticity,  $\xi$ , of 2.0 one can apply the result of Ref. 1 to calculate the growth time of the head-tail instability driven by the impedance of the kicker magnets. The resulting  $\tau$  is about 30 sec. The effect of the octupole field generated by 12 spool octupoles carrying a current of 0.35 Amps and located at  $\beta = 98$  m is described by  $G/\pi = 2.67 \times 10^4 \text{ m}^{-1}$ . This combined with a smaller contribution to the octupole field coming from the dipoles and given by<sup>2</sup>  $G/\pi = 2.55 \times 10^3 \text{ m}^{-1}$  leads, according to Eq. (10), to the emittance growth of  $\dot{\epsilon} \sim 10^{-13} \text{ m rad sec.}^{-1}$ , which is equivalent to about  $2\pi \text{ mm mrad/hour}$ .

Finally, I would like to thank Dave Finley, Gerry Jackson, Chuck Ankenbrandt and Mike Harrison for stimulating discussions.

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<sup>2</sup>"Amplitude Dependence of the Tune Shift"- N.M. Gelfand, 1987 IEEE Particle Accelerator Conference Proceedings, p1014